Math 10A with Professor Stankova
Worksheet, Discussion #7; Monday, 9/11/2017
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Chain Rule

Example

1. Find $\frac{d}{dx} f(g(x))$.

Solution: The chain rule tells us that this derivative is $f'(g(x)) \cdot g'(x)$.

Problems

2. Find $(\cos(x^2))'$.

Solution:

$$\frac{d}{dx}(\cos(x^2)) = -\sin(x^2) \cdot \frac{d}{dx}x^2 = -2x\sin(x^2).$$

3. Find the derivative of $(\tan x)^2$.

Solution:

$$\frac{d}{dx}(\tan x)^2 = 2\tan x \cdot \frac{d}{dx}\tan x = 2\tan x \sec^2 x.$$

4. Find the derivative of $\dfrac{x}{1-\sin x}$.

Solution: This is actually an application of the quotient rule:

$$\frac{d}{dx}\frac{x}{1-\sin x} = \frac{(1-\sin x)\cdot \frac{d}{dx}x - x\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2} = \frac{1-\sin x - x(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{1-\sin x + x\cos x}{(1-\sin x)^2}.$$

5. Find the derivative of $\ln(\sin x)$.

Solution:
$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\cdot\frac{d}{dx}\sin x = \frac{\cos x}{\sin x}.$$

6. Find the derivative of $\sec(x)$.

Solution:
$$\frac{d}{dx}\sec(x) = \frac{d}{dx}(\cos x)^{-1} = -(\cos x)^{-2}\cdot\frac{d}{dx}\cos x = \frac{\sin x}{\cos^2 x}.$$

7. Find the derivative of $\sin(\cos x)$.

Solution:
$$\frac{d}{dx}\sin(\cos x) = \cos(\cos x)\cdot\frac{d}{dx}\cos x = -\cos(\cos x)\sin x.$$

8. Find the derivative of $e^{\sin(2x)}$.

Solution:
$$\frac{d}{dx}e^{\sin(2x)} = e^{\sin(2x)}\cdot\frac{d}{dx}\sin(2x) = e^{\sin(2x)}\cdot\cos(2x)\cdot\frac{d}{dx}(2x) = 2\cos(2x)e^{\sin(2x)}.$$

9. Find the derivative of $\tan(e^{\sin x})$.

Solution:
$$\frac{d}{dx}\tan(e^{\sin x}) = \sec^2(e^{\sin x})\cdot\frac{d}{dx}e^{\sin x} = \sec^2(e^{\sin x})\cdot e^{\sin x}\cdot\frac{d}{dx}\sin x$$
$$= \sec^2(e^{\sin x})\cdot e^{\sin x}\cdot\cos x.$$

10. Find the derivative of $\cos(\tan(3x))$.

Solution:

$$\frac{d}{dx}\cos(\tan(3x)) = -\sin(\tan(3x)) \cdot \frac{d}{dx}\tan(3x) = -\sin(\tan(3x)) \cdot \sec^2(3x) \cdot \frac{d}{dx}3x$$
$$= -3\sin(\tan(3x)) \cdot \sec^2(3x).$$

11. Find the derivative of $((2x+3)^5 + e^x)^{99}$.

Solution:

$$\frac{d}{dx}((2x+3)^5 + e^x)^{99} = 99((2x+3)^5 + e^x)^{98} \cdot \frac{d}{dx}((2x+3)^5 + e^x)$$
$$= 99((2x+3)^5 + e^x)^{98} \cdot \left(\frac{d}{dx}(2x+3)^5 + \frac{d}{dx}e^x\right)$$
$$= 99((2x+3)^5 + e^x)^{98} \cdot [5(2x+3)^4 \cdot 2 + e^x].$$
$$= 99((2x+3)^5 + e^x)^{98} \cdot [10(2x+3)^4 + e^x].$$

12. Find the derivative of $\arctan(\cos x)$.

Solution:
$$\frac{d}{dx}\arctan(\cos x) = \frac{1}{1+\cos^2 x} \cdot \frac{d}{dx}\cos x = \frac{-\sin x}{1+\cos^2 x}.$$

13. Find the derivative of $\tan(\arctan(x))$.

Solution: Note that $\tan(\arctan(x)) = x$ since $\tan, \arctan$ are inverses and so the derivative is just 1. Using the chain rule though, this is

$$\frac{d}{dx}\tan(\arctan(x)) = \sec^2(\arctan(x)) \cdot \frac{d}{dx}\arctan(x) = \frac{\sec^2(\arctan(x))}{1+x^2}.$$

Note that $\sec^2(x) = \frac{1}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = 1 + \tan^2(x)$ and hence

$$\frac{\sec^2(\arctan(x))}{1+x^2} = \frac{1+\tan^2(\arctan(x))}{1+x^2} = \frac{1+x^2}{1+x^2} = 1.$$

Derivative of Inverse Functions

Example

14. Find the derivative of $f^{-1}(x)$.

Solution: Since we know that $f(f^{-1}(x)) = x$, taking the derivative of both sides and using the chain rule gives us

$$f'(f^{-1}(x)) \cdot \frac{df^{-1}}{dx}(x) = 1 \implies \frac{df^{-1}}{dx}(x) = \frac{1}{f'(f^{-1}(x))}.$$

Problems

15. Let $f(x) = x^3 + 7x + 2$. Find the tangent line to $f^{-1}(x)$ at $(10, 1)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(10)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(10) = \frac{1}{f'(f^{-1}(10))} = \frac{1}{f'(1)} = \frac{1}{10}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 1 = \frac{x - 10}{10} \implies y = \frac{x}{10}.$$

16. Let $f(x) = x^5 + 3x^3 + 7x + 2$. Find the tangent line to $f^{-1}(x)$ at $(13, 1)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(13)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(13) = \frac{1}{f'(f^{-1}(13))} = \frac{1}{f'(1)} = \frac{1}{21}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 1 = \frac{x - 13}{21} \implies y = \frac{x + 8}{21}.$$

17. Let $f(x) = e^{-2x} - 9x^3 + 4$. Find the tangent line to $f^{-1}(x)$ at $(5, 0)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(5)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(0)} = \frac{-1}{2}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 0 = \frac{x-5}{-22} \implies y = \frac{-x+5}{2}.$$

18. Let $f(x) = x^7 + 2x + 9$. Find the tangent line to $f^{-1}(x)$ at $(12, 1)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(12)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(12) = \frac{1}{f'(f^{-1}(12))} = \frac{1}{f'(1)} = \frac{1}{9}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 1 = \frac{x-12}{9} \implies y = \frac{x}{9} - \frac{1}{3}.$$

19. Let $f(x) = x^{5/3}e^{x^2}$. Find the tangent line to $f^{-1}(x)$ at $(e, 1)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(e)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(e) = \frac{1}{f'(f^{-1}(e))} = \frac{1}{f'(1)} = \frac{3}{11e}.$$

We found this by using the product rule to find the derivative of f since

$$f'(x) = \frac{5}{3}x^{2/3}e^{x^2} + x^{5/3} \cdot e^{x^2} \cdot 2x.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 1 = \frac{3(x-e)}{11e} \implies y = \frac{3x}{11e} + \frac{8}{11}.$$

20. Let $f(x) = \dfrac{-e^{-3x}}{x^2+1}$. Find the tangent line to $f^{-1}(x)$ at $(-1, 0)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(-1)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(-1) = \frac{1}{f'(f^{-1}(-1))} = \frac{1}{f'(0)} = \frac{1}{3}.$$

We found this by using the quotient rule to find the derivative of f since

$$f'(x) = \frac{(x^2+1)\cdot(-e^{-3x})\cdot(-3) - (-e^{-3x})\cdot(2x)}{(1+x^2)^2}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 0 = \frac{x-(-1)}{3} \implies y = \frac{x+1}{3}.$$

21. Let $f(x) = 7x + \sin(2x)$. Find the tangent line to $f^{-1}(x)$ at $(0, 0)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(0)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(0)} = \frac{1}{9}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 1 = \frac{x-0}{9} \implies y = \frac{x}{9}.$$

22. Let $f(x) = x^3 + 8x + \cos(3x)$. Find the tangent line to $f^{-1}(x)$ at $(1, 0)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(1)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{8}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 0 = \frac{x-1}{8} \implies y = \frac{x-1}{8}.$$

23. Let $f(x) = 10x + (\arctan(x))^2$. Find the tangent line to $f^{-1}(x)$ at $(0, 0)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(0)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(0)} = \frac{1}{10}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 0 = \frac{x-0}{10} \implies y = \frac{x}{10}.$$

24. Let $f(x) = 7x^3 + (\ln x)^3$. Find the tangent line to $f^{-1}(x)$ at $(7, 1)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(7)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(7) = \frac{1}{f'(f^{-1}(7))} = \frac{1}{f'(1)} = \frac{1}{21}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 1 = \frac{x-7}{21} \implies y = \frac{x}{21} + \frac{2}{3}.$$

25. Let $f(x) = x^3 + x - 2$. Find the tangent line to $f^{-1}(x)$ at $(0, 1)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(0)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(1)} = \frac{1}{4}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 1 = \frac{x-0}{4} \implies y = \frac{x}{4} + 1.$$

26. Let $f(x) = x^3 + 2x - 8$. Find the tangent line to $f^{-1}(x)$ at $(4, 2)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(4)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{14}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 2 = \frac{x-4}{14} \implies y = \frac{x}{14} + \frac{12}{7}.$$